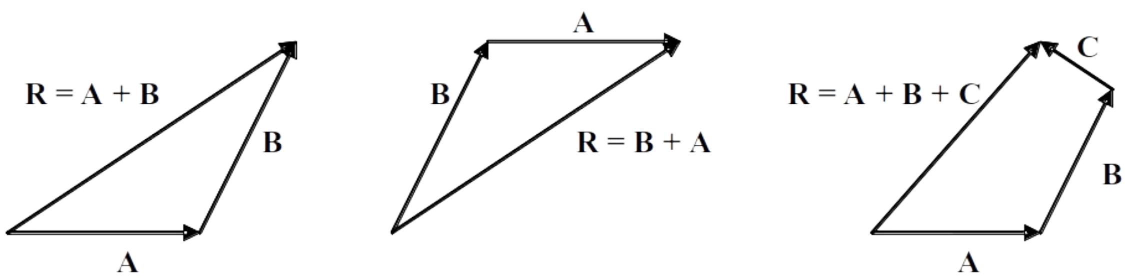
Motion in Two Dimensions

Vector review

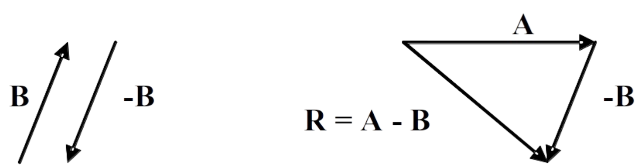
A *vector* quantity can be represented as an arrow pointing in a direction. It has magnitude and direction. Examples: displacement, velocity, force.
A *scalar* quantity has magnitude but no direction. Examples: distance, speed, mass, temperature.

Addition of vectors ***A*** and ***B***

R = ***A*** + ***B*** (= ***B*** + ***A***)

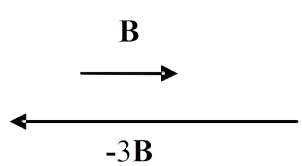


Subtraction of vectors: ***R*** = ***A*** – ***B*** = ***A*** + (-***B***)



Multiplication of a scalar *a* times a vector:

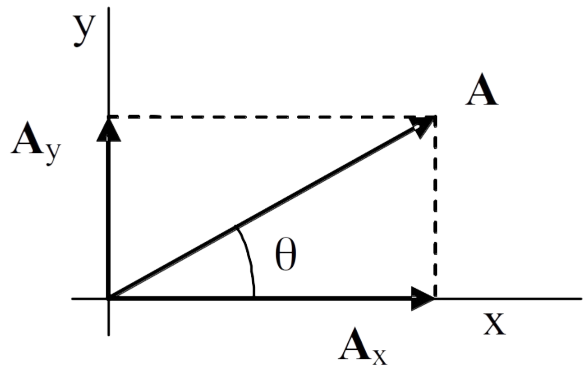
C = *a**B***　　　　　　Example: ***C*** = -3***B***



Components of a vector

In Cartesian coordinates, a vector ***A*** can be resolved into its x- and y- components.

$$A = A_x + A_y$$
$$A_x = A\cos\theta\,,\ \ A_y = A\sin\theta$$
$$A = \sqrt{A_{x^2} + A_{y^2}}$$
$$\tan\theta = A_y / A_x$$



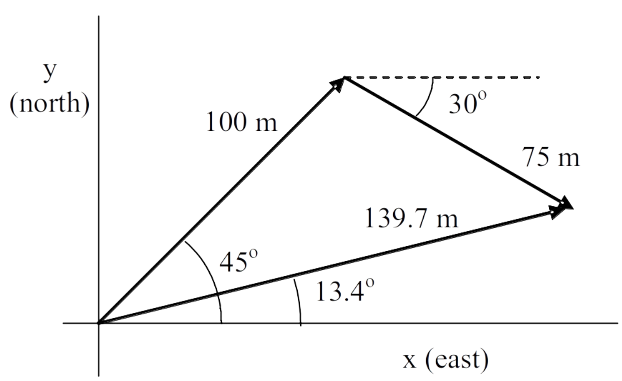
Adding vectors using components:

$$R = A + B$$
$$R_x = A_x + B_x$$
$$R_y = A_y + B_y$$
$$R = \sqrt{R_x^2 + R_y^2}$$
$$\tan\theta = R_y / R_x$$

Example

A boy walks 100 m 45° north of east, then walks 75 m 30° south of east. What is his net displacement (magnitude and direction)?

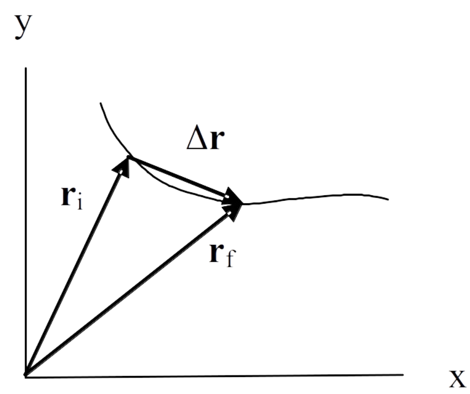
$$A = 100\ m\ @\ 45^o$$
$$B = 75\ m\ @\ -30^o$$
$$A_x = 100\ m\ \cos(45^o) = 70.7\ m$$
$$A_y = 100\ m\ \sin(45^o) = 70.7\ m$$
$$B_x = 75\ m\ \cos(-30^o) = 65.0\ m$$
$$B_y = 75\ m\ \sin(-30^o) = -37.5\ m$$
$$R_x = 70.7\ m + 65.0\ m = 135.7\ m$$
$$R_y = 70.7\ m - 37.5\ m = 33.2\ m$$
$$R = \sqrt{(135.7)^2 + (33.2)^2} = 139.7\ m$$
$$\theta = \tan^{-1}(R_y / R_x)$$
$$= \tan^{-1}(33.2/135.7) = 13.4^o$$



Displacement, velocity, and acceleration in 2-D

Displacement

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$



Average velocity

$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t}$$

Instantaneous velocity (calculus only)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The velocity is always in a direction tangent to the path.

Average acceleration

$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t}$$

Instantaneous acceleration (calculus only)

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

The acceleration can be in any direction (see chapter on Circular Motion)

Projectile motion

The acceleration is constant and has only the vertical component due to gravity. There is no acceleration in the *x* direction

x-direction:

$\Delta x = v_{x0} t$

$v_x = v_{x0} = const$

y-direction:

$\Delta y = v_{y0} t - \frac{1}{2} g t^2$

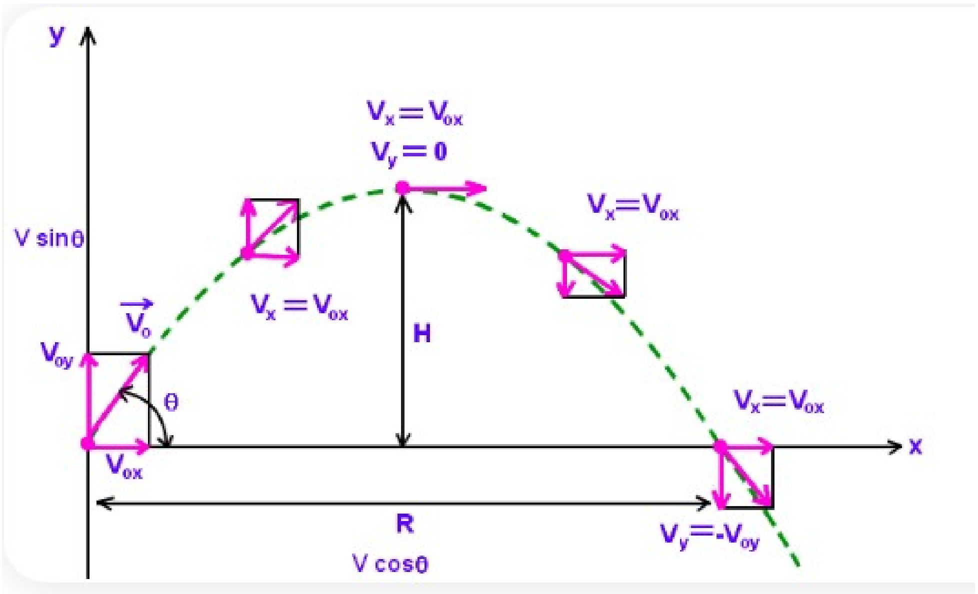
$v_y = v_{y0} - gt$

$v_y^2 = v_{y0}^2 - 2 g \Delta y$

total velocity:

$v = \sqrt{v_x{}^2 + v_y{}^2}$

$\theta = \tan^{-1}(v_y / v_x)$



Example

- Time of flight of a projectile over flat ground.

$$\Delta y = v_{y0} t - \frac{1}{2} g t^2 = 0 = t \left(v_{y0} - \frac{1}{2} g t \right) \Rightarrow t = 0 \ (start) \ or \ t = \frac{2 v_{y0}}{g} \quad (finish)$$

- Range of a projectile over flat ground.

$$R(v_0, \theta) = \Delta x = v_{x0} t = \frac{2 v_{x0} v_{y0}}{g} = \frac{2 v_0^2 \sin\theta_0 \cos\theta_0}{g} = 2 v_0^2 \sin(2\theta)$$

- Maximum height of a projectile. At peak height

$$v_y = v_{y0} - gt = 0 \Rightarrow t = \frac{v_{y0}}{g}$$

$$v_y = v_{y0} - gt = 0, \ t = \frac{v_{y0}}{g}, \ \Delta y = v_{y0} t - \frac{1}{2} g t^2 = v_{y0} \left(\frac{v_{y0}}{g} \right) - \frac{1}{2} g \left(\frac{v_{y0}}{g} \right)^2$$

$$\Delta y = \frac{(v_{y0})^2}{2g}$$

Example

A football is kicked from the ground with an initial speed of 25 m/s at an angle of 30º above the ground.

$$v_{x0} = v_0 \cos\theta_0 = (25\ m/s)\ \cos(30^o) = 21.7\ m/s$$
$$v_{y0} = v_0 \sin\theta_0 = (25\ m/s)\ \sin(30^o) = 12.5\ m/s$$
$$hang\ \ time\ = \frac{2 v_{y0}}{g} = \frac{2(12.5)}{9.8} = 2.55\ s$$
$$range\ = v_{x0} t = (21.7)(2.55) = 55.4\ m$$
$$maximum\, elevation\ = \frac{v_{y0}^2}{2g} = \frac{(12.5)^2}{2(9.8)} = 8.0\ m$$

What is the speed and direction of motion of the all 2 s after it is kicked?

$$v_x = v_{x0} = 21.7\ m/s,\ \ v_y = v_{y0} - gt = 12.5 - (9.8)(2) = -7.1\ m/s$$
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(21.7)^2 + (-7.1)^2} = 22.8\ m/s$$
$$\theta = \tan^{-1}(v_y / v_x) = \tan^{-1}(-7.1/21.7) = -18.1^o$$

Example

A projectile is fired from the edge of the roof of a building 20 m tall with an initial speed of 25 m/s at an angle of 30º above the horizon.

- How long does it take for the projectile to reach the ground below?

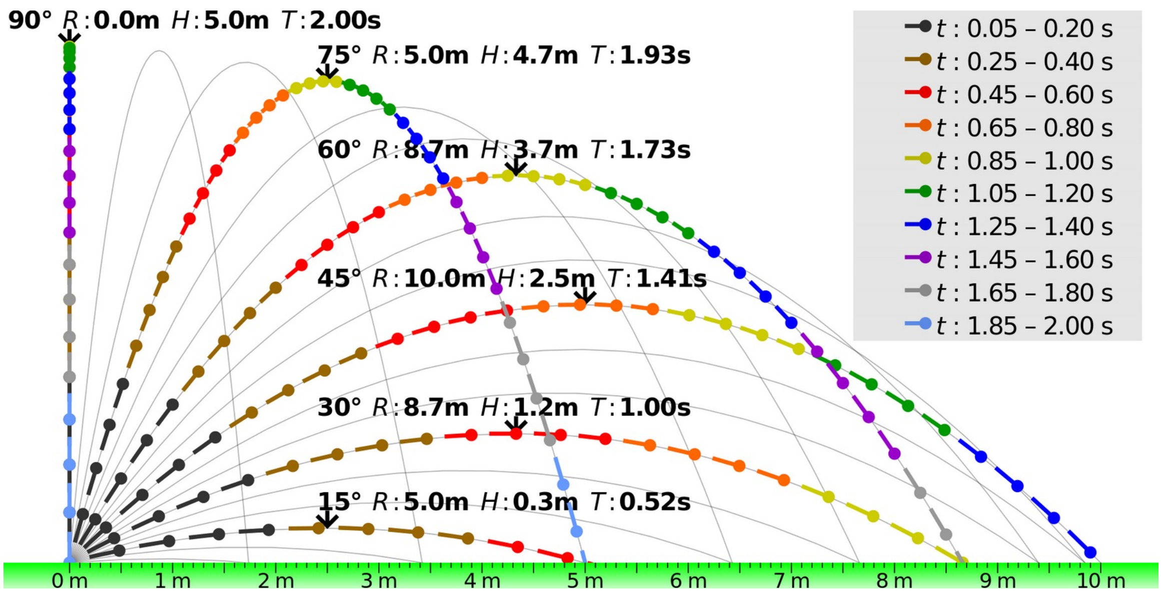
$$\Delta y = v_{y0} t - \tfrac{1}{2} g t^2$$
$$-20 = 25(\sin(30^o) t - \tfrac{1}{2}(9.8) t^2 = 12.5 t - 4.9 t^2$$
$$4.9 t^2 - 12.5 t - 20 = 0\ \ (of\ \ form\ \ at^2 + bt + c = 0)$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12.5 \pm \sqrt{12.5^2 - 4(4.9)(-20)}}{2(4.9)} = \frac{12.5 \pm 23.4}{9.8} = \underline{3.66\ s},\ -1.1\ 0\ s$$

- What is the speed of the projectile just before it lands?

$$v_x = v_{x0} = v_0 \cos\theta_0 = 25 \cos(30^o) = 21.7\ m/s$$
$$v_y = v_{y0} - gt = 25 \sin(30^o) - 9.8(3.66) = -\ 23.4\ m/s$$
$$v = |(21.7)^2 + (-23.4)^2| = 31.9\ m/s$$

For further thought:

- How far does the projectile land from the base of the building?
- What would be the final speed if the projectile were thrown straight down with a speed of 25 m/s instead of being up at an angle of 30º?
- What is the meaning of the negative time in the solution?
- Can you solve for the final speed without using the quadratic formula?
- Can you solve for the time of flight without using the quadratic formula?

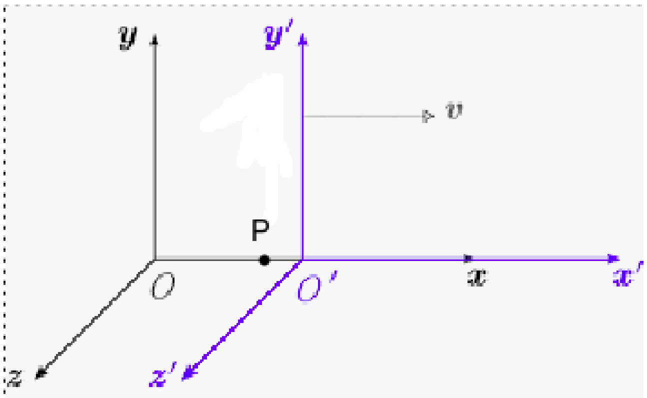


Relative Motion

An observer is said to be a *inertial* observer if it moves at constant velocity (with v = zero being a special case). A more precise definition will be given using the concept of force. Supposed the observers O is at rest and O' moves along the x-axis at constant velocity v

Relative position

The point P has coordinates (x,y,z) for O and coordinates P' (x',y',z') for O'. The transformations which relate the two coordinates systems are the Galileo transformations:



$x' = x - vt$
$y' = y$
$z' = z$

Example

The two observers Alice and Tom are at a train station. Alice is sitting on a bench located at (0,0,0) while Tom is sitting on a train located at (0,0,0). Both of them are at rest (not moving). There is a bicycle located at P = (3,0,0).
At t = 0 the train starts to move with speed v_T = 2 m/s. Find the coordinates P' of the bicycle as seen by Tom.

Since Tom is on the train, he sees the bicycle moving. For him the coordinates of the bicycle change in time as

$x'(t) = x - 2t$

t	0	1	2	3	4
x'	3	1	-1	-3	-5

The figure above corresponds to t = 2: the train is located at $x = v_T \Delta t$ = 4 m and the position of the bike is $x'(t=2) = 3 - 2\cdot2 = -1$ m

Relative velocity one dimension

Now Bob gets on the bike and starts to ride with velocity v_B respect to Alice. What is the velocity of Bob, v_B' , as observed by Tom on moving train?

the velocity v_B' given by

$$v_B' = \frac{\Delta x'_B}{\Delta t} = \frac{\Delta x_B}{\Delta t} - \frac{\Delta x_T}{\Delta t} = v_B - v_T$$

where Δx_B and Δx_T are the displacements of the bike and of the train respect to Alice and $\Delta x'_B$ is the displacement of the bike respect to Tom.

Example

If Bob on the bike rides at velocity $v_B = v_T$, what is his velocity respect to Tom?

Since the bike is moving as fast as the train, Tom seems the bike always at the same distance away, i.e. the Bob is not moving respect to Tom.

$$v_B' = v_B - v_T = 0$$

Example

If Bob on the bike rides with speed v_B = 1 m/s in opposite direction of the train, what is his velocity respect to Tom?

$$v_B' = v_B - v_T = (-1) - (+2) = -3x\ m/s$$

Since the bicycle moves in the opposite direction of the train, for Tom the bicycle is moving faster than as observed by Alice.

Relative velocity two dimensions

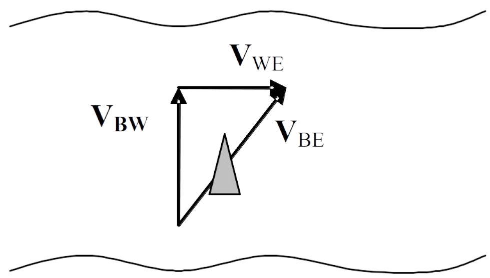
The velocity of an object A depends on the coordinate system in which it is measured. If two systems, B and C, are moving with respect to each other, then velocities transform as

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

Example A boat can travel with a speed of 4 m/s in still water.

- If the boat heads straight across a river flowing at 3 m/s, what is the speed of the boat with respect to the earth?

$$v_{BW} = 4\ m/s,\ v_{WE} = 3\ m/s$$
$$v_{BE} = v_{BW} + v_{WE}\ (vector\ \ addition)$$



Since the boat and water velocities are perpendicular,

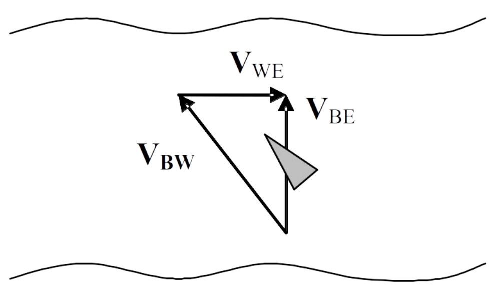
$$v_{BE} = \sqrt{4^2 + 3^2} = 5\ m/s$$

If the river is 50 m across, how far downstream does the boat land on the other side of the river?

$$t = \frac{y}{v_{BW}} = \frac{50}{4} = 12.5\ s,\ \ x = v_{WE}\, t = (3)(12.5) = 37.5\ m$$

- In what direction should the nose of the boat be pointed so that it goes straight across the river?

$$\theta = \tan\left(\frac{v_{WE}}{v_{BE}}\right) = \tan\left(\frac{3}{4}\right) = 36.9^o$$
$$(upstream)$$



- What is the speed of the boat with respect to the water?

$$v_{BE} = \sqrt{v_{BW^2} - v_{WE^2}} = \sqrt{4^2 - 3^2} = 2.65\ m/s$$

- How long does it take the boat to cross the river?

$$t = \frac{y}{v_{BE}} = \frac{50}{2.65} = 18.9\ s$$